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CONTROL OF THE TEMPERATURE REGIME IN A LAYER OF HEAT-CONDUCTING MATERIAL

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The article presents the solution of the problem of finding the optimum controlling heat flux on the boundary of a plane heat-conducting plate ensuring stabilization of the temperature in the specified section with known disturbing heat flux on the other boundary.

We are concerned with an infinite plane plate with thickness d, with the heat flux z(t) being specified on one of its boundaries. We have to find such a heat flux u(t) on the other boundary that in the specified section $x_0 \in [0, d]$ the regularity of change of temperature y(t) is ensured.

In dimensionless form the problem is described by the one-dimensional heat-conduction equation

$$\frac{\partial \Theta}{\partial t} = \frac{\partial^2 \Theta}{\partial x^2} \tag{1}$$

with boundary conditions of the second kind

$$\left.\frac{\partial \Theta}{\partial x}\right|_{x=0} = z(t); \left.\frac{\partial \Theta}{\partial x}\right|_{x=\pi} = u(t), \ t \geqslant 0, \tag{2}$$

the initial condition

$$\Theta|_{t<0} = \Theta_0(x) \tag{3}$$

and the condition

$$\Theta_{x=x_0} = y(t), \ t \geqslant 0, \ x_0 \in [0, \pi]. \tag{4}$$

In a fairly similar statement Kuznetsov [1] investigated the problem of stabilization $\Theta(x_0, t)$ without disturbing effect z(t). Stated somewhat similarly, the authors of [2, 3]

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also examined the problem but it was impossible to modify the methods of solution presented by the authors to apply to the problem in question.

Most similar to the problem under examination is the approach of [4-6] where from the value $\Theta(x_0, t)$ measured in x_0 the heat flux $u(t)$ has to be determined on condition that the heat flux $z(t)$ is known or accurately measured. These authors explained in detail the method of solving linear as well as nonlinear [7] problems of this type. These problems may be classed as identification of functional dependences inaccessible to measurement, and their solution is complicated primarily because of the errors of measurement and the errors of numerical solution, reducing such problems to Hadamard's ill-posed problems [8]. It is important here that when there are no errors of measurement, the functions $\Theta(x_0, t)$ and $z(t)$ always belong to the class of those functions for which the solution $u(t)$ of Eq. (1) exists unambiguously and belongs to the class of physically realizable functions.

The problem solved in the present work differs first of all by the fact that in the general case $z(t)$ belongs to the class of functions for which control of $u(t)$ need not exist. In that case we must speak of such a $u(t)$ which would ensure the minimum of some selected functional.

A problem similar to the stated one was also examined in [9] where in principle the authors reduced it to the previously mentioned problems of identification, but they solved it in distinction to [4] by a more laborious and less universal method.

Applying the cosine transform to both sides of Eq. (1), we obtain

$$\frac{\partial}{\partial t}\Theta_{nc}(t) = \frac{2}{\pi}[(-1)^n u(t) - z(t)] - n^2\Theta_{nc}(t), \tag{5}$$

where

$$\Theta_{nc}(t) = \frac{2}{\pi}\int_0^{\pi}\Theta(x, t)\cos(nx)\,dx.$$

We take the Laplace transform from expression (5), and expressing $\Theta_{nc}(p)$ through the other parameters, we obtain:

$$\Theta_{nc}(p) = \frac{(-1)^n u(p) - z(p) + \Theta_{nc}(0)}{p + n^2}. \tag{6}$$

The cosine series for $\Theta(x, p)$ can be written via the values of $\Theta_{nc}(p)$ in the form

$$\Theta(x, p) = \frac{\Theta_{0c}(p)}{2} + \sum_{n=1}^{\infty}\Theta_{nc}(p)\cos(nx) = \frac{\sqrt{\pi/p}}{2\,\mathrm{sh}(\pi\sqrt{p})}[u(p)\,\mathrm{ch}(x\sqrt{p}) - z(p)\,\mathrm{ch}[(\pi - x)\sqrt{p}]] + \Psi(p),$$

where

$$\Psi(p) = \frac{\Theta_{0c}(0)}{2p} + \sum_{n=1}^{\infty}\frac{\Theta_{nc}(0)}{p + n^2}\cos(nx).$$

We now express the boundary control $u(p)$ through $y(p)$, $z(p)$, and $\Psi(p)$, taking condition (4) into account:

$$u(p) = \frac{1}{\mathrm{ch}(x_0\sqrt{p})}\left[z(p)\,\mathrm{ch}[(\pi - x_0)\sqrt{p}] + \frac{y(p) - \Psi(p)}{\sqrt{\pi/p}}\,\mathrm{sh}(\pi\sqrt{p})\right]. \tag{7}$$

If $\Theta_0(x) = \Theta_0 = \text{const}$, then $\Psi(p) = \Theta_0/p$, and the introduction of the new function $\overset{\circ}{y}(t) = y(t) - \Theta_0$ leads to the exclusion of the initial distribution Θ_0. Henceforth we will take it that $\overset{\circ}{y}(t) \equiv 0$, i.e.,

$$u(p) = z(p)\,\frac{\mathrm{ch}[(\pi - x)\sqrt{p}]}{\mathrm{ch}(x\sqrt{p})} = z(p)K_z(x_0, p). \tag{8}$$

The solution of Eq. (8) in the domain of the original ensures that the temperature $\Theta(x_0, t) \equiv 0$ is maintained. Thus, the problem of stabilizing $\Theta(x_0, t)$ reduces to the determination of the inverse Laplace transform from expression (8). For $x_0 \in [\pi/2, \pi]$ this problem is unambiguously solvable and requires either inversion of expression (8), if z(p) is specified in the domain of the transform, or calculation of the integral of the convolution

$$u(t) = \int_0^t z(\tau) K_z(x_0, t-\tau)\, d\tau, \; x_0 \in [\pi/2, \pi], \tag{9}$$

if z(t) is specified in the domain of the original.

The Green function $K_z(x_0, t)$ is expressed analytically from $K_z(x_0, p)$ [10] by the formula

$$K_z(x_0, t) = L^{-1}\left[\frac{\operatorname{ch}[(\pi - x_0)\sqrt{p}]}{\operatorname{ch}(x_0\sqrt{p})}\right] = L^{-1}\left[\frac{\operatorname{ch}[(a-y)\sqrt{p}]}{\operatorname{ch}(a\sqrt{p})}\right] = -\frac{1}{a}\frac{\partial}{\partial y}\vartheta_2\left(\frac{y}{2a}, \frac{i\pi t}{a^2}\right)\Big|_{0<y<2a}, \tag{10}$$

where $\vartheta_2(x, \tau) = 2\sum_{k=0}^{\infty} e^{i\pi\tau(k+0.5)^2}\cos((2k+1)\pi x)$ is Jacobi's theta function [11]. On the basis of (10) we obtain for $K_z(y_0, t)$:

$$K_z(x_0, t) = \frac{\pi}{x_0^2}\sum_{k=0}^{\infty}(2k+1)\sin\left(\frac{\pi^2}{2x_0}(2k+1)\right)e^{-\left(\frac{\pi}{2x_0}(2k+1)\right)^2 t}; \quad x_0 \in \left[\frac{\pi}{2}, \pi\right]. \tag{11}$$

In the special case for $x_0 = \pi/2 - K_z(x_0, t) \equiv 1$, i.e., $u(t) \equiv z(t)$. For $x_0 < \pi/2$ an inverse transform of $K_z(x_0, p)$ does not exist, and in such a case u(t) can be determined from the solution of Volterra's integral equation of the first kind with the kernel $K(t, \tau) = K(t-\tau)$ [12]:

$$z(t) = \int_0^t K_z(\pi - x_0, t-\tau)\, u(\tau)\, d\tau = A(x_0, u(t)), \; x_0 \in [0, \pi/2), \tag{12}$$

where $K_z(\pi - x_0, t-\tau)$ is the kernel of the integral transform corresponding to expression (11). It is known [8, 12, 13] that the equation in question requires regularizing algorithm [14] to be advised. It was pointed out above that in the general case the heat flux z(t) belongs to the class of functions for which a solution $u(t) \in C$ of Eq. (12) does not exist. In this case we may speak of a solution u(t) for Eq. (12) only when the problem is stated in variational form for minimization of the quadratic functional

$$\int_0^{T+\tau} [A(x_0, u(t)) - z^*(t-\tau)]^2 dt = \min, \tag{13}$$

where

$$z^*(y) = \begin{cases} z(y), & y \geqslant 0, \\ 0 & y < 0; \end{cases}$$

τ is the delay of the controlling flux u(t) relative to the "zero" of the flux z(t).

The existence of the delay τ in the functional (13) must not be viewed as a purely artificial shift of the disturbance z(t) along the time axis by the time $\tau^* > t_{opt}$, and a new disturbance $z^*(t)$ cannot be taken, for which with $t < \tau^*$ $z^*(t) \equiv 0$. In the investigated class of problems the delay τ^* has to be infinite in principle for attaining the absolute "minimum" of the functional (13); in this sense it is in general analogous to the regularization parameter in inverse problems where the "best" solution of the problem is attained when it is equal to zero. Insignificant deviations of the functional (13) for $\tau > \tau_{opt}$ have to correspond to the optimum value τ_{opt} whereas insignificant deviations of the form of the controlling action u(t) for $\tau < \tau_{opt}$ have to correspond to τ_{opt}. In the present work τ_{opt} was chosen in particular on the basis of such considerations.

We will minimize the functional (13) for u(t) belonging to the class of functions such that

$$u_{\min}(t) \leqslant u(t) \leqslant u_{\max}(t). \tag{14}$$

We point out that for the correctness of the problem of minimizing the functional (13), condition (14) is indispensable. In the interval [0, T] we take a uniform grid with the number of divisions n_T and a step $\Delta T = T/n_T$. We determine the number of divisions for the time of delay τ by the expression

$$n_\tau = \left[\frac{\tau}{\Delta t}\right] + 1, \tag{15}$$

and we denote the total number of divisions in the interval $[0, T + \tau]$ by $N = n_T + n_\tau$. We will seek the solution u(t) in the form of piecewise-constant functions, such that

$$u_i = u\,(t \in (\Delta t\,(i-1),\ \Delta t i)).$$

We will approximate the integrals contained in (13) by quadrature formulas using the values in the i-th nodes of the grid. As a result of obtain a problem of minimization in quadratic form

$$\sum_{i=1}^{N} a_i \left[\sum_{j=1}^{i} K_j u_{i-j+1} - z_i^*\right]^2 = \min, \tag{16}$$

where

$$K_j = \int_{(j-1)\Delta t}^{j\Delta t} K(\tau)\,d\tau,\quad z_i^* = z^*(t_i - \tau) = \begin{cases} z_{i-n\tau}, & i \geqslant n_\tau, \\ 0, & i < n_\tau, \end{cases}$$

$$a_i = \begin{cases} 1, & i \neq 1,\ N, \\ 1/2, & i = 1,\ N. \end{cases}$$

The functional (16) may be written in matrix form:

$$1/2U^T A^* U - U^T B = \min, \tag{17}$$

where

$$A_{ij}^* = A_{ji}^* = \sum_{n=j-i+1}^{N-i+1} a_{n+i-1} K_n K_{n+i-j},\ i \geqslant j,$$

$$B_i = \sum_{n=i}^{N} a_{n+i-1} K_{n-i+1} z_n^*,\ U^T = [u_1,\ u_2,\ u_3,\ \ldots,\ u_N].$$

The matrix A* in the quadratic form (17) is positively determined [15]; this follows from a comparison of (17) and (16). However, for a certain kind of dependence of the kernel K(t) the matrix A* may be close to degenerate, and the problem of minimizing the quadratic form (17) will be ill-posed because of errors of calculation in the algorithm for minimization. For regularization of problem (17) we use the Tikhonov stabilizer $\Omega_n(u(t))$ for $n = 0$ ensuring weak regularization [13], and with it taken into account, the quadratic form of (17) then appears as

$$1/2U^T A U - U^T B = \min, \tag{18}$$

where $A = A^* + \alpha E$, α is the parameter of weak regularization. The quadratic form of (18) has to be minimized with respect to the vector U with the constraints

$$U_{\min} \leqslant U \leqslant U_{\max}, \tag{19}$$

ensuing from the equality (14), where

$$U_{\min}^T = [u_{1\min}, u_{2\min}, \ldots, u_{N\min}],\ U_{\max}^T = [u_{1\max}, u_{2\max}, \ldots, u_{N\max}].$$

The parameter α is chosen on the basis of the requirement of ensuring positive determinacy of the matrix A in the count.

The problem of minimizing the quadratic form (18) with constraints type (19) and a positively determined matrix A is a problem of quadratic programming [16] which can be solved by the method of Lagrange multipliers [15]. The Lagrange function for (18) has the form

$$L(U, \lambda) = \frac{1}{2} U^T AU - U^T B + \lambda^T (D - CU), \tag{20}$$

where λ is the vector of the Lagrange multipliers, $\lambda_i \geqslant 0$; D, C are obtained from the constraints (19): $D^T = [U_{\min}^T - U_{\max}^T]$, $C^T = [E, -E]$, E is the unique diagonal matrix of dimensionality N.

The solution of problem (18) is the saddle point (U^*, λ^*) of the Lagrange function (20) satisfying the system of equations [16]:

$$CU^* \geqslant D;\ \lambda^* \geqslant 0;\ AU^* - B = C^T\lambda^* \tag{21}$$

with the conditions of complementariness

$$\lambda_i^* ((CU^*)_i - D_i) = 0,\ i = 1, 2, \ldots, N.$$

The algorithm for seeking the saddle point of the Lagrange function in the given problem is greatly simplified because of the singular types of matrices D, C, and it reduces to the algorithm presented below.

We assume that in U we distinguish two sets: the set of "removals" U^-, consisting of the components of the vector U for which

$$u_i = (u_{i\max} \cup u_{i\min}), \tag{22}$$

and the set of "inclusions" U^+ consisting of the components remaining in U after formation of the set U^-, for which

$$u_{i\min} < u_i < u_{i\max}. \tag{23}$$

Expressions (22), (23) are analogous to the conditions of complementariness of the system (21).

The algorithm for seeking the saddle point consists in the successive verification whether the sets U^+ and U^- contain certain conditions. For the set U^+ we find the solution $\overline{U}^+$ of the system of equations

$$A^+\overline{U}^+ = B^+, \tag{24}$$

where the matrices A^+, B^+ were obtained from A, B by removing the rows corresponding to the set U^-. After determining $\overline{U}^+$, we find the vector δ each of whose components is determined by the expression

$$\delta_i = \left(\frac{u_{i\max} - u_i^+}{\overline{u}_i^+ - u_i^+} : \overline{u}_i^+ > u_i^+\right) \cup \left(\frac{u_i^+ - u_{i\min}}{u_i^+ - \overline{u}_i^+} : \overline{u}_i^+ \leqslant u_i^+\right).$$

If among the components of the vector δ there are values $\delta_i < 1$, the value $\varepsilon_j = \min_i \delta_i$ is also sought, and to obtain the new set U^+ the following relaxation is effected:

$$U^+ := (\overline{U}^+ - U^+)\,\varepsilon_j + U^+,$$

whereupon the element j is transferred from the set U^+ to the set U^-, and for the new set U^+ the system of equations (24) is again solved.

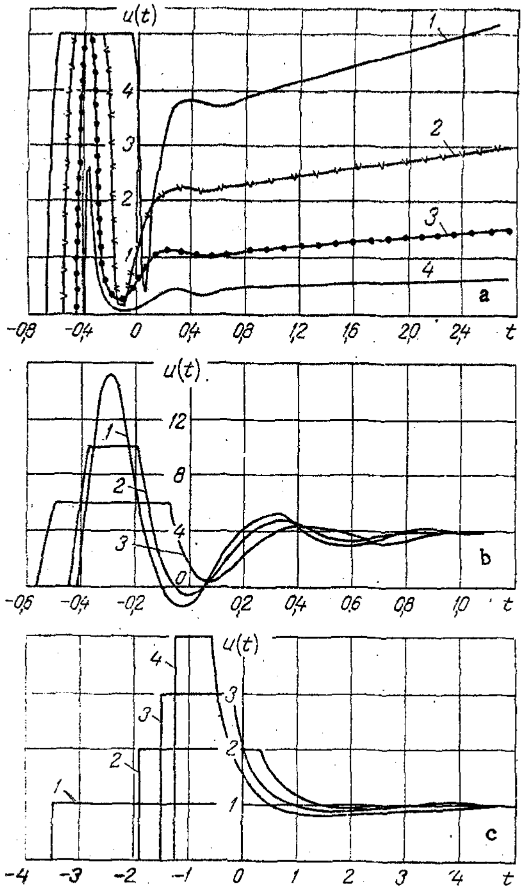

Fig. 1. Time dependences of the optimal controlling fluxes u(t): a) $z(t) = \alpha t$ for α equal to: 1) 0.68; 2) 0.4; 3) 0.2; 4) 0.08; b) $z(t) = 0.68t$ with different values of u_{max}: 1) 16; 2) 10; 3) 6; c) $z(t) = 1$ for different u_{max}: 1) 1; 2) 2; 3) 3; 4) 4.

When the condition $\forall\delta_i \geqslant 1$ for elements of the vector δ is fulfilled, the set U^- is tested according to the conditions

$$(A^-U)_i < B_i : u_i^- = u_{i\max},$$
$$(A^-U)_i \geqslant B_i : u_i^- = u_{i\min}, \quad (25)$$

which correspond to the third equation of system (21). If the conditions (25) are not fulfilled for the subscript i, then the corresponding i-th element is transferred from U^- to U^+, and then the set U^+ is analyzed. When conditions (25) are fulfilled for $\forall u_i$, then the obtained sets U^+ and U^- are the solution of the problem of minimization of the quadratic form (18) with the constraints (19).

The presented algorithm is distinguished by the simplicity of its realization and highly rapid effect. Like with any problem of regularization, the efficiency of the algorithm depends on the correct selection of the parameter α of the Tikhonov stabilizer $\Omega_0(u(t))$.

The realization of the presented algorithm for minimizing the functional (13) with the constraints (14) was effected in FORTRAN for a BESM-6 computer. With the aid of the program the controlling flux u(t) with disturbing flux z(t), corresponding to the expression $z(t) = \alpha t$, $t \geqslant 0$, for different values of α, was sought.

The constraints (14) for u(t) were taken for $u_{min}(t) = u_{min} = \text{const}$ and $u_{max}(t) = u_{max} = \text{const}$. The graphs of the dependence u(t) for $x_0 = 0$, $\alpha = \text{var}$, $u_{min} = -5$, $u_{max} = +5$ are shown in Fig. 1a, and of the dependence u(t) for $a = 0.68$, $u_{min} = -5$, $x_0 = 0$, u_{max} = var in Fig. 1b.

It can be seen from the presented graphs that for u(t) there takes place a forced regime in which "acceleration" of the system and then its "braking" is ensured. In the forced regime the shape of u(t) is decisively affected by the values u_{max}, u_{min} on which the magnitude of the delay τ and the discrepancy $||A(x_0, u(t)) - z(t)||_C$ depend; for the specified z(t) the maximum discrepancy occurs for t = 0. We point out that for the problem under examination, there may be no forced regime only when the disturbance z(t) is specified such that for it all derivatives up to the infinite one exist on the segment t = 0. It should be expected that the forced regime will be hardest when z(t) has discontinuities of the first kind, and in particular, when the disturbance z(t) changes jumplike from zero to z_{max} in such a way that $z(t) = z_{max}$ for $t \geqslant 0$. It is characteristic that for similar disturbances we have to determine most often only the quasioptimal control of u(t) ensuring that

$$\|A(x_0, u(t)) - z(t)\|_{L_2} \leqslant \gamma \tag{26}$$

on condition that

$$\|u(t+\tau) - z(t)\|_C = \min, \tag{27}$$

and that the time of delay τ in principle is not bounded. In Fig. 1c there are illustrated several controls u(t) for z(t) = 1, t < 0, distinguished by the constraint of u_{max}. The selection of the corresponding regularity u(t) depends on the value of γ in the condition (26), and then (27) is automatically fulfilled. We point out that with large values of γ, the control u(t) repeats the regularity of the change z(t) with the delay τ (curve 4 in Fig. 1c).

The suggested algorithm, and most importantly, the devised program may be used for a large range of problems of similar type without any substantial alterations. In particular, Eq. (1) may be replaced by a similar equation in cylindrical coordinates, and also the boundary conditions (2) may be of the first or third kind. At the same time the Green function $K_z(x_0, t)$, which is analogous to (10), can be obtained both analytically and by the known numerical methods.

NOTATION

Θ, temperature; L^{-1}, inverse Laplace transform; A (...), linear transform; L(u, λ), Lagrange function; λ, vector of the Lagrange multipliers; Ω_n, Tikhonov stabilizer; α, parameter of weak regularization.

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DURATION OF THE FREEZING OF BODIES WITH VARIABLE TEMPERATURE OF THE MEDIUM

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UDC 536.2

The article contains an analysis of the application of the integral method of thermal moments of zeroth order in determining the duration of freezing of bodies with simple shape when the temperatures of the cooling medium is variable.

Approximate analytical solutions of unidimensional Stefan-type problems for determining the duration of processes of nonsteady heat conduction are conveniently found by using the so-called integral methods [1]. To these also belongs the method of thermal moments of zeroth order [2]. The application of this method to problems with phase transformations at constant temperature of the medium was studied, e.g., in [3, 4]. The essence of the method is that the initial integral relation is obtained as a result of integrating the principal differential equation of heat conduction twice with respect to the space coordinate and once with respect to time. Into this relation we then substitute the equations of the temperature distribution profiles (invariant to shifts of the front of phase transformation) and the regularity of change of the cooling (heating) impulse on the surface of the body, determined as the area in coordinates temperature vs time between the lines of temperature change at the end of the investigated region (body).

The method of thermal moments of zeroth order may also be applied to determining the time of motion of the fronts of phase transformation in bodies of simple shape when the temperature of the medium is variable. Although it is expedient to use the integral statement of the problem [4], we demonstrate below how to obtain the initial integral relation of the thermal moments from the differential statement of the problem because the method itself is relatively little known.

Let us examine the problem of the cooling of bodies with simple shape (sphere, unbounded cylinder, and plate) with phase transformations

$$C(T)\,\omega(x)\,\frac{\partial T(x,\ \tau)}{\partial \tau} = \frac{\partial}{\partial x}\left(\lambda(T)\,\omega(x)\,\frac{\partial T(x,\ \tau)}{\partial x}\right),\ 0 \leqslant x \leqslant l; \tag{1}$$

$$T(x,\ 0) = T_0(x); \tag{2}$$

$$\frac{\partial T(0,\ \tau)}{\partial x} = 0; \tag{3}$$

$$\alpha(\tau)\,(T(l,\ \tau) - T_c(\tau)) = -\lambda(T(l,\ \tau))\,\frac{\partial T(l,\ \tau)}{\partial x}; \tag{4}$$

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